q-PD thickenings and q-PD envelopes

a deformation

prism. cohom $\leftrightsquigarrow$ q-crystalline cohom $\leftrightsquigarrow$ q-de Rham cohom

shows indep. of coordinates of q-de Rham cohom

Gives explicit complexes to compute $\Delta_{R^{(1)}/A}$ like de Rham-Witt complex computes crystalline cohom.

<u>Today</u>: Foundations for q-crystalline cohom

Need deformed versions of PD-thick. and PD-envelopes to define it and write down Čech-Alexander complexes (like before)

Preliminaries

- $A := \mathbb{Z}_p[\![q-1]\!] := \mathbb{Z}_p[q]^{\wedge}_{(q-1)}$ throughout the talk

  $\mathbb{Z}_p$ is an A-alg via $A \longrightarrow \mathbb{Z}_p$, $q \longmapsto 1$.

- For $n \in \mathbb{N}$, set

  $$[n] := [n]_q := \frac{q^n - 1}{q-1} = 1 + q + \dots + q^{n-1} \in A.$$

  Have $[n] \longmapsto n$ under $A \longrightarrow \mathbb{Z}_p$. q-analogue of $n \in \mathbb{N} \subset \mathbb{Z}_p$

- Have $A/_{[p]} \cong \mathbb{Z}_p[\zeta_p] \subset \mathbb{Q}_p(\zeta_p)$

  $q + ([p]) \longmapsto \zeta_p$

  where ζ_p is a primitive p-th root of 1.

- A is a δ-ring via extension of δ on $\mathbb{Z}_p$ $\left(\delta(n) = \frac{n - n^p}{p}\right)$ by $\delta(q) := 0$ (i.e. $\phi(q) := q^p$), and $A \longrightarrow \mathbb{Z}_p$ is a δ-map.

Observation

(i) $[p] \equiv p \mod (q-1)$ in A.

(ii) $[p] \in A$ is distinguished.

(iii) $(p, q-1)$ and $(p, [p])$ define the same top. on A.

(iv) $(A, [p])$ is a bounded prism.

(v) $(A, q-1)$ is not a prism.

Proof:

(i) As $A \xrightarrow{q \mapsto 1} A/_{q-1} = \mathbb{Z}_p$.

$[p] \longmapsto p$

(ii) $A \longrightarrow \mathbb{Z}_p$ a δ-map

$\overset{(i)}{\Longrightarrow} \delta([p]) \equiv \delta(p) = 1 - p^{p-1} \mod (q-1)$

$\Longrightarrow \delta([p]) = 1 + x$, for some $x \in (p, q-1) = \mathrm{Rad}(A)$

$\Longrightarrow \delta([p]) \in A^\times$.

(iii) • $(p, [p]) \overset{(i)}{\subset} (p, q-1)$

• In $\mathbb{Z}_p[\zeta_p]$: $p = u(\zeta_p - 1)^{p-1}$, for some $u \in \mathbb{Z}_p[\zeta_p]^\times$.

$\Longrightarrow$ In $A/_{(p,[p])} \cong \mathbb{Z}_p[\zeta_p]/_{(p)}$: $(q-1)^{p-1} = u^{-1}p = 0$

$\Longrightarrow (p, q-1)^{p-1} \subset (p, (q-1)^{p-1}) \subset (p, [p])$.

(iv) By (ii) and $A/_{[p]} \cong \mathbb{Z}_p[\zeta_p]$ p-torsionfree

[then in part bounded p^∞-torsion

(v) Assume $q-1$ distinguished

$\Longrightarrow$ image under $A \to \mathbb{Z}_p$ distinguished

↯ as this is 0. $\square$

<u>Def</u>: A <u>q-PD pair</u> is a derived $(p,[p])$-complete δ-pair (D,I) over $(A,q-1)$ (i.e. a δ-A-alg D s.t. $q-1\in I$) s.t.

(1) $\phi(I)\subset [p]D$ and $\gamma(I)\subset I$ where

$$\gamma(x) := \frac{\phi(x)}{[p]} - \delta(x), \text{ for } x\in I.$$

- makes sense because D is $[p]$-torsionfree by (2).
- Use this notation whenever it makes sense

(2) $(D,[p])$ is a bounded prism over $(A,[p])$, i.e. D is $[p]$-torsionfree and $D/[p]$ has bounded p^∞-torsion.

(3) $D/q-1$ is p-torsionfree

[and with finite $(p,[p])$-complete Tor-amplitude /D]

Call such $D\to D/I$ a <u>q-PD thickening</u>.

<u>Examples</u>:

(1) $(A,q-1)$ is a q-PD pair.

- $\phi(q-1)=\phi(q)-\phi(1)=q^p-1=(q-1)(q^{p-1}+\dots+q+1)$
$\qquad = (q-1)[p]$

- $\gamma(q-1)=(q-1)-\delta(q-1)$
$\rightsquigarrow$ As $\mathbb{Z}_p\cong A/q-1$ is p-torsionfree, suffices to show $p\,\delta(q-1)\in(q-1)$:
$p\,\delta(q-1)=\phi(q-1)-(q-1)^p=(q-1)([p]-(q-1)^{p-1})$.

More generally:

Let D be a $(p,[p])$-cpltly flat A-alg, then $(D, q-1)$ is a q-PD pair.

A bit more specifically again:

Let $(S, \square)$ be a framed pair, i.e.

- a form. smooth $\mathbb{Z}_p$-alg S
- with a form. étale hom.

$$\square : \mathbb{Z}_p[x_1, \dots x_n]^{\wedge(p)} \longrightarrow S$$

(a framing "↭ choice of local coordinates")

Set $D := S[\![q-1]\!]$. One shows

- $\square$ deforms uniquely to a $(p,q-1)$-cpltly étale

$$\hat{\square} : \mathbb{Z}_p[q-1, x_1, \dots, x_n]^{\wedge(p,q-1)} \longrightarrow S[\![q-1]\!].$$

- $S[\![q-1]\!]$ is flat$/A$.

$\rightsquigarrow (D, q-1)$ is a q-PD pair.

(2) ($q=1$ case)

Let D be a p-tf, p-cplt δ-ring.

$\rightsquigarrow$ via $A \xrightarrow{q \mapsto 1} \mathbb{Z}_p \longrightarrow D$ an A-alg.

Let $I \subset D$ be a p-cplt ideal. Then

(D, I) is a q-PD pair $\Leftrightarrow$ $\forall x \in I, n \geq 0: \frac{x^n}{n!} \in I$

Suffices to show:

$$\phi(I) \subset pD, \ \gamma(I) \subset I \iff \text{RHS}$$

"$\Leftarrow$" For $x \in I$

- $\phi(x) = x^p + p\delta(x) = p\left((p-1)! \dfrac{x^p}{p!} + \delta(x)\right) \in pD$

  as $\frac{x^p}{p!} \in D$.

- $\gamma(x) = \dfrac{x^p}{p} = (p-1)! \dfrac{x^p}{p!} \in I$

"$\Rightarrow$" For $x \in I$:

- $\dfrac{x^p}{p!} = \underbrace{\dfrac{1}{(p-1)!}}_{\in D^\times} \cdot \dfrac{x^p}{p} = \dfrac{1}{(p-1)!} \gamma(x) \in I$

- Do induction. Use for $n = k \cdot p$:

$$\frac{x^n}{n!} = \underbrace{\frac{k!(p!)^k}{(k p)!}} \cdot \Bigg(\underbrace{\frac{1}{k!}\left(\frac{x^p}{p!}\right)^k}_{\in I \text{ by induction}} \Bigg)$$

$$v(\ldots) \overset{\text{Legendre}}{=} \sum_{i \geq 1} \left\lfloor \frac{k}{p^i} \right\rfloor + k - \sum_{i \geq 1} \left\lfloor \frac{kp}{p^i} \right\rfloor$$

$$= \sum_{i \geq 1} \left\lfloor \frac{k}{p^i} \right\rfloor - \sum_{j \geq 0} \left\lfloor \frac{k}{p^j} \right\rfloor + k = 0$$

so a well-def elt in D.

(3) (A perfect example)

Let B be a perfect $(p,[p])$-complete δ-A-alg.

(e.g. the $(p,[p])$-completion of the perfection of A).

Then $(B, \xi := \phi^{-1}([p]))$ is a q-PD pair.

- $\phi(\xi) \in [p]B$ clear
- $\gamma(\xi) = \frac{\phi(\xi)}{[p]} - \delta(\xi) = \frac{[p]}{[p]} - \delta(\xi)$

  $\Rightarrow$ suffices to show $\delta(\xi) \equiv 1 \mod \xi$

  B perfect p-complete δ-ring $\Rightarrow$ disting. elements are non-zerodiv.

  $\Rightarrow$ suffices to show $p\delta(\xi) \equiv p \mod \xi$

  But
  - $p\delta(\xi) = \phi(\xi) - \xi^p \equiv \phi(\xi) \mod \xi$
  - $\phi(\xi) = [p] \equiv p \mod (q-1)$
  - $\xi \mid (q-1)$ as $q - 1 = \frac{q-1}{q^{\frac{1}{p}}-1}(q^{\frac{1}{p}}-1) = \xi \cdot \phi^{-1}(q-1)$.
- $(B,[p])$ perfect prism $\overset{\text{Andreas}}{\underset{\text{talk}}{\Longrightarrow}}$ bounded
- B already p-torsionfree

Lemma 1:

Let D be a $[p]$-torsionfree, $(p,[p])$-cplt δ-A-alg. Then $\phi^{-1}([p]D)$ is stable under γ, i.e. for $x \in D$:

$$\phi(x) \in [p]D \implies \phi(\gamma(x)) \in [p]D.$$

Rmk: For $q=1$, the lemma gives:

Let D be a p-torsionfree, p-cplt δ-ring. Then, for $x \in D$:
$$x^p \in pD \implies x^{p^2} \in p^{p+1}D.$$

$\ulcorner$ $x^p \in pD$

$\phi(\_) \equiv (\_)^p \bmod pD$

$$\implies \phi(x) \in pD$$

$$\overset{\text{Lemma}}{\implies} \phi\left(\frac{\phi(x)}{p} - \delta(x)\right) \in pD$$

$$\implies \left(\frac{x^p}{p}\right)^p = \left(\frac{\phi(x)}{p} - \delta(x)\right)^p \in pD$$

$$\implies x^{p^2} \in p^{p+1}D$$

$\llcorner$

Have seen this already in Mattia's talk.

Prop. 2:

Let

$$\begin{tikzcd} A \arrow[r, "\text{form. smooth}"] \arrow[d] & P \arrow[d, two heads] \arrow[dl, phantom, "\text{2}"] \\ \mathbb{Z}_p \arrow[r, "\text{form. smooth}"'] & R \end{tikzcd}$$

$\longleftarrow$ δ-A-alg

$J := \mathrm{Ker}(P \longrightarrow R)$

and assume

(*) $J/_{q-1} \subset P/_{q-1}$ is gen. by $x_1 \ldots x_r \in J$ which form a reg. seq. in $P/_{(p,q-1)}$.

Then there is a univ. map $(P, J) \longrightarrow (D, I)$ of δ-pairs with (D, I) a q-PD pair. Moreover:

(1) D is $(p, [p]_q)$-cplbly flat $/A$.

(2) Get $P/_J \xrightarrow{\sim} D/_I$ so that $D \longrightarrow D/_I \cong R$ is a q-PD thick. of R.

(3) $D/_{q-1} \cong (\text{PD-env. of } P/_{q-1} \to R)^{\wedge(p)}$ $\longleftarrow$ From Mathia's talk

Rmk: • Call $D_{J,q}(P) := P$ the <u>q-PD envelope</u> of (P, J).

• If $q = 1$ in P, (3) implies
$$D_{J,1}(P) = (\text{PD-env of } J \text{ in } P)^{\wedge(p)}$$

• Always have:
$(P, J) =$ filt. colimit of (P_i, J_i) with P_i noeth., J_i satisfies (*) locally on $\mathrm{Spec}(P_i)$.

Γ $\rightsquigarrow (P, J) := (P_i, J_i)$, J fin. gen.

$$\begin{tikzcd} \mathbb{Z}_p \arrow[r, "\text{form. smooth}"] \arrow[dr, "\text{form smooth}"'] & P/_{q-1} =: P' \arrow[d, two heads] \\ & R \end{tikzcd}$$

$J' := \mathrm{Ker}(P' \longrightarrow R)$

- $\mathbb{Z}_p \xrightarrow[\text{smooth}]{\text{form.}} P' \overset{\text{Tag 031J}}{\Longrightarrow} \Omega^1_{P'/\mathbb{Z}_p}$ proj P'-mod.
- $\mathbb{Z}_p \xrightarrow[\text{smooth}]{\text{form.}} R \overset{\text{"}}{\Longrightarrow} 0 \to J'/J'^2 \to \Omega^1_{P'/\mathbb{Z}_p} \otimes_{P'} R \to \Omega^1_{R/\mathbb{Z}_p} \to 0$

split exact.

$\Rightarrow J'/J'^2$ proj R-mod

J fin. gen.
$\Rightarrow J'/J'^2$ finite free after loc.

Nakayama
$\Rightarrow$ after loc: $\exists x_1, \dots, x_r \in J'$: $\cdot\ J' = (x_1, \dots, x_r)$
$\cdot\ x_1, \dots, x_r$ give R-basis of J'/J'^2.

$\Rightarrow x_1, \dots, x_r$ q-reg seq in P', i.e. $R[T_1, \dots, T_r] \xrightarrow{\sim} \bigoplus_{n \geq 0} J'^n / J'^{n+1}$

$\Rightarrow$ after loc: $x_1, \dots, x_r$ is a reg seq in P'

Partial proof of Prop. 2:

Set
$$D := P\left\{\frac{\phi(x_1)}{[p]_q}, \dots, \frac{\phi(x_r)}{[p]_q}\right\}^{\wedge}_{(p, [p]_q)}$$

ad (1): $x_1, \ldots, x_r$ reg seq. in $P/(p, q-1)$

$$\Leftrightarrow \underset{\substack{\| \\ \phi(x_1) - \underbrace{p\delta(x_1)}_{=0}}}{x_1^p}, \ldots, x_r^p \quad \text{—"—}$$

For $J' := ([p], \phi(x_1), \ldots, \phi(x_r))$, can apply last corollary in Mattia's talk to (P, J') over $(A, [p])$. Get:

- $D = P\left\{\frac{J'}{[p]}\right\}^{\wedge(p,[p])}$ is $(p, [p])$-cplltly flat$/(A, [p])$.
- Compatibility with base change $A \to \mathbb{Z}_p$, i.e.
$$D/q-1 \cong P/q-1\left\{\frac{\phi(x_1)}{p}, \ldots, \frac{\phi(x_r)}{p}\right\}^{\wedge(p)}.$$

ad(3): • $P/q-1$ is p-cplltly flat $/\mathbb{Z}_p$

$\ulcorner$ $A \to P$ form. smooth $\Rightarrow \ldots \Rightarrow$ flat

$\Rightarrow A/q-1 = \mathbb{Z}_p \longrightarrow P/q-1$ flat

$\llcorner$ In part., $P/q-1$ is p-cplltly flat $/\mathbb{Z}_p$

- $\operatorname{Ker}\left(P/q-1 \to R\right) = (x_1, \ldots, x_r)$ with $x_1, \ldots, x_r$ reg seq in $(P/q-1)/p$.

$\Rightarrow$ Can apply statement after Thm 3 in Mattia's talk.

$$\left(\text{PD-env of } P/q-1 \longrightarrow R\right)^{\wedge(p)}$$
$$\cong P/q-1\left\{\frac{\phi(x_1)}{p}, \ldots, \frac{\phi(x_r)}{p}\right\}^{\wedge(p)} \cong D/q-1.$$

ad (2): This gives $D \xrightarrow{\pi} D/_{q-1} \xrightarrow{\tau} R$,

let $I \subset D$ be its kernel, i.e.

$$I = \pi^{-1}\,\mathrm{Ker}(\tau).$$

In part., $P \longrightarrow D \longrightarrow R$ induces $P/_J \xrightarrow{\sim} D/_I$.

$\Rightarrow D \longrightarrow D/_I \cong R$ is a q-PD thick.
(when (D, I) is a q-PD pair).

(D, I) is a q-PD pair:

Only non-triv assertions to check: $\phi(I) \subset [p]_q D$,
$\gamma(I) \subset I$.

Set
$$I' := \phi^{-1}([p]D) \cap I.$$

wts: $\gamma(I') \subset I'$, $I = I'$.

- By construction: $q-1, x_1, \dots, x_r \in I'$.

$$\phi(q-1) = \dots = (q-1)[p] \in [p]D$$
$$\phi(x_i) = [p] \cdot \frac{\phi(x_i)}{[p]} \in [p]D$$

- I' is stable under $\gamma(\_)$.

Let $x \in I'$.
  - Lemma 1 $\Rightarrow \gamma(x) \in \phi^{-1}([p]D)$
  - $\pi(\gamma(x)) = \frac{\pi(x)^p}{p}$ and $\pi(x) \in \mathrm{Ker}(\tau)$.

  But τ is a PD-thick, so $\mathrm{Ker}(\tau)$ has divided powers
  $$\Rightarrow \frac{\pi(x)^p}{p} \in \mathrm{Ker}(\tau)$$
  $$\Rightarrow \gamma(x) \in \pi^{-1}(\mathrm{Ker}(\tau)) = I$$

- As I, I' are $(p,[p])$-sylt., it suffices to show $I'/_{q-1} \longrightarrow I/_{q-1}$ surj.

⌈ $I'/_{q-1} \longrightarrow I/_{q-1}$ surj

$\Rightarrow I'/_{(p,q-1)} \longrightarrow I/_{(p,q-1)}$ surj

$\overset{I', I \text{ are}}{\underset{(p,q-1)\text{-sylt}}{\Longrightarrow}} I' \longrightarrow I$ surj

⌊

- But via (3): $I/_{q-1} = \operatorname{Ker}(\tau) \subset I'/_{q-1}$

⌈ [Proof of Stacks Proj, Tag 07GS]:

Since τ PD-thick, $\operatorname{Ker}(\tau)$ is the smallest p-sylt. ideal of $D/_{q-1}$ which contains the gen. $x_1, \dots, x_r$ of $\operatorname{Ker}(P/_{q-1} \to R)$ and is stable under $\overline{\gamma} : x \mapsto \frac{x^p}{p}$.

We saw: $I'/_{q-1}$ satisfies this.

⌊

<u>Universality</u>: Omitted

□

<u>Example</u> (q-PD envelope of $\Delta(\mathbb{A}^1) \subset \mathbb{A}^1 \times \mathbb{A}^1$)

Let $P := \mathbb{Z}_p[q, x, y]^{\wedge (p, q-1)}$, δ-ring via $\delta(q) = \delta(x) = \delta(y) = 0$.

Let $J := (x - y) \subset P$ (ideal of diagonal in $\mathbb{A}^1 \times \mathbb{A}^1$).

<u>Claim:</u> $D_{J,q}(P) \overset{\text{Def}}{=} P\{\frac{x^p - y^p}{[p]_q}\}^{\wedge (p, [p])}$ is top. free $/\mathbb{Z}_p[q, x]$

with basis $\gamma_{k,q}(x-y) := \dfrac{(x-y)(x-qy)\cdots(x-q^{k-1}y)}{[k]_q!}$, $k \geq 0$

where $[k]_q! := \prod_{i=1}^{k} [i]_q$. (analog. top. free $/\mathbb{Z}_q[q, y]$)

See [Pridham, "On q-de Rham coh...", 1.5].

Here only: $\gamma_{p,q}(x-y) \in D_{J,q}(P)$:

- As $[k]_q \in A^\times$, for $k < p$, suffices to show:

  $[p] \mid (x-y)(x-qy)\cdots(x-q^{p-1}y)$ in $D_{J,q}(P)$

- $P \xrightarrow{q \mapsto \zeta_p} \mathbb{Z}_p[\zeta_p, x, y]^{\wedge (p)}$ with kernel $[p]P$, and

$$\prod_{i=0}^{p-1} (x - \zeta_p^i y) = x^p - y^p \quad \text{in } \mathbb{Z}_p[\zeta_p, x, y]^{\wedge (p)}.$$

In $\mathbb{Z}_p[\zeta_p]$: $z^p - 1 = \prod_{i=0}^{p-1} (z - \zeta_p^i)$

$\Rightarrow (x-y)(x-qy)\cdots(x-q^{p-1}y) \equiv x^p - y^p \mod [p]P$

But $[p] \mid x^p - y^p = [p] \dfrac{x^p - y^p}{[p]}$ in $D_{J,q}(P)$.